Summary term 2024

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Elliptic curves and their moduli spaces

Announcements

(1) This is the last problem sheet. Its points are bonus points.

(2) July 15/16 will be the last problem sessions. It will also be an opportunity to ask review the course contents. Simply send your questions to Adam, Saverio or me by email. We will exchange and collect them before the problem session.

(3) The first exam will take place on August 5, 9–11 a.m. in the large lecture hall Wegelerstr. 10. It will be a written exam. The style of questions will be a bit similar to questions asked during an oral exam – more information will be given during the last lecture (July 17).

Bonus homework problems (due July 12)

Problem 1 (Universal property of the j-invariant)

(a) Let $\mathcal{W}$ and $\mathcal{E}\ell\ell$ denote the Weierstrass and elliptic curves functor from the lecture. Denote by $\mathcal{E}\ell\ell[1/6]$ the restriction to schemes over $\mathbb{Z}[1/6]$. Let $\mathbb{A}^1_{\mathbb{Z}[1/6]} = \mathbb{G}_m\backslash\mathcal{W}$ denote the categorial quotient from Proposition 13.14 (the j-line). Show that there is a unique natural transformation $\mathcal{J} : \mathcal{E}\ell\ell \to \mathbb{G}_m\backslash\mathcal{W}$ that fits into the diagram

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{E}\ell\ell[1/6] \\ {\scriptstyle j}\downarrow & \swarrow {\scriptstyle \exists!\, \mathcal{J}} & \\ \mathbb{A}^1_{\mathbb{Z}[1/6]}. & & \end{array}$$

Hint: Assume E is an elliptic curve over S and $6 \in \mathcal{O}_S(S)^\times$. The line bundle ω_E is locally trivial, so locally E comes from $\mathcal{W}$. Glue this into a construction of a point $\mathcal{J}(S)$.

(b) Show that $\mathcal{J}$ has the following universal property. Let T be any $\mathbb{Z}[1/6]$-scheme. Then every natural transformation $\mathcal{F} : \mathcal{E}\ell\ell[1/6] \to T$ factors uniqly through $\mathcal{J}$:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{a} & \mathcal{E}\ell\ell[1/6] \\ {\scriptstyle j}\downarrow & \swarrow {\scriptstyle \mathcal{J}} & \downarrow {\scriptstyle \mathcal{F}} \\ \mathbb{A}^1_{\mathbb{Z}[1/6]} & \xrightarrow[\exists!]{} & T. \end{array}$$

Hint: The composition $\mathcal{F} \circ a : \mathcal{W} \to T$ is $\mathbb{G}_m$-invariant and hence factors (uniquely) through the categorical quotient.

Problem 2 (Rings of invariants)

(a) Let $\mu_{2,k}$ act on $\mathbb{A}^2_k$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ (Yoneda sense). Determine the coaction map $\mu^* : k[X, Y] \to k[t]/(t^2 - 1) \otimes_k k[X, Y]$ and the invariants $k[X, Y]^{\mu_{2,k}}$.

(b) Let k be of characteristic p. Let $\alpha_{p,k} = \operatorname{Spec} k[t]/(t^p)$ with group scheme structure $m^*(t) = t \otimes 1 + 1 \otimes t$. Consider the action

$$\mu : \alpha_{p,k} \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k, \quad \mu^*(X) = t \otimes 1 + 1 \otimes X.$$

Determine the invariants $k[X]^{\alpha_{p,k}}$.