Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due May 3)

Problem 1 (Rigidity for abelian varieties)

Abelian varieties have a rigidity property that is analogous to that of $\mathbb{G}_m$ (see Proposition 2.12 in the lecture notes). More precisely, the following statement is true:

Let A_1, A_2 be two abelian varieties over k and let Y be a connected k-scheme that has a rational point $x : \operatorname{Spec} k \to Y$. Then restriction to the fiber over x defines an isomorphism

$$\operatorname{Hom}_{Y\text{-}group}(Y \times_{\operatorname{Spec} k} A_1, Y \times_{\operatorname{Spec} k} A_2) \xrightarrow{\sim} \operatorname{Hom}_{k\text{-}group}(A_1, A_2)$$
$$f \longmapsto f|_{\{x\}\times A_1}.$$

The general proof is a bit tricky, but show the following special cases with the methods from the lecture:

(a) The case $k = \bar{k}$ and Y integral of finit type.

(b) The case that x is the closed point of $Y = \operatorname{Spec} A$ for a local ring A.

Problem 2 (Generic smoothness)

(a) Let K/k be a finite field extension. Prove that $\Omega^1_{K/k} = 0$ if and only if K/k is separable.

(b) Let K/k be a finitely generated field extension. That is, K is the field of fractions of a finite type k-algebra. Assuming that $\operatorname{char}(k) = 0$, prove that $\dim_K \Omega^1_{K/k}$ is equal to the transcendence degree of K/k.

Hint: Noether normalization and the "Kähler differential arithmetic" from the lecture.

(c) Let X/k be of finite type and integral. Assume that $\operatorname{char}(k) = 0$. Deduce from (b) that there exists a dense open subscheme $U \subseteq X$ that is smooth over k.

Further Problems

Problem 3 (Group schemes in characteristic 0 are smooth)

Let k be an algebraically closed field of characteristic 0 and let G be a reduced finite type group scheme over k. Using (c) of the previous problem and a translation argument, show that G is smooth.

Problem 4 (Kähler differentials)

Let A be an R-algebra. Determine the function $\operatorname{Spec}(A) \ni x \mapsto \dim_{\kappa(x)} \Omega^1_{A/R}(x)$ in the following two cases:

(a) $R = \mathbb{Z}$ and $A = \mathbb{Z}[\zeta_3]$, where ζ_3 is a primitive third root of unity.

(b) $R = \mathbb{C}$ and $A = \mathbb{C}[X, Y, Z]/(XY, XZ, YZ)$.