Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Homework problems (due May 15)

Problem 1 (Meromorphic functions)

Prove Lemma 6.7 from the lecture notes: Let C, C_1, C_2 be curves over k and let t denote the variable on $\mathbb{P}^1_k$.

(1) For every non-constant meromorphic function f on C, there exists a unique non-constant morphism $\varphi : C \to \mathbb{P}^1_k$ such that $\varphi^*(t) = f$. This defines a bijection

$$\kappa(\eta) \setminus k \xrightarrow{\sim} \mathrm{Mor}_k(C, \mathbb{P}^1_k) \setminus k.$$

(2) Let $\varphi : C_1 \to C_2$ be a non-constant morphism and let $f_2 \in \kappa(\eta_2)$ be a meromorphic function on C_2. Then

$$\mathrm{div}(\varphi^*(f_2)) = \varphi^*(\mathrm{div}(f_2)).$$

Problem 2 (Properties of divisors)

Prove Proposition 6.6 from the lecture notes:

(1) Let $\varphi : C_1 \to C_2$ be a non-constant morphism of curves over k and let $D_i \in \mathrm{Div}(C_i)$. Then

$$\deg(\varphi_*(D_1)) = \deg(D_1), \quad \deg(\varphi^*(D_2)) = \deg(\varphi) \cdot \deg(D_2), \quad \varphi_*(\varphi^*(D_2)) = \deg(\varphi) \cdot D_2.$$

(2) Let f be a meromorphic function on C. Then $\deg(\mathrm{div}(f)) = 0$.

(See the lecture notes for hints.)

Further Problems

Problem 3 (Two concrete morphisms)

Let t be the coordinate function on $\mathbb{P}^1_{\mathbb{Q}}$, let φ be one of the two morphisms

$$t(t^2+1), \ \frac{t}{(t-1)^2} : \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}},$$

and let $x \in \{1, \infty\}$. Determine the integers $\deg(\varphi)$, e_x and f_x, as well as the divisors $\varphi_*([x])$ and $\varphi^*([x])$.