HOMEWORK #9 IN ALGEBRAIC STRUCTURES 2

1. Prove Lemma 10.1.

2. Prove Lemma 10.2.

3. Prove Lemma 10.3.

4) Let $L := \mathbb{Q}(\sqrt{-2}), N_{L/\mathbb{Q}} : L^* \to \mathbb{Q}^*$ the norm map.
a) Describe the image of $N_{L/\mathbb{Q}}$,
b) show that $x = a^2 + 2b^2, y = c^2 + 2d^2, a, b, c, d \in \mathbb{Z}$ then we can find $u, v \in \mathbb{Z}$ such that $xy = u^2 + 2v^2$.

5) Let K be a field $p = \mathrm{ch}(K) > 0, L \supset K$ a Galois extension such that $[L : K] = p$. Show that there exists $l \in L - K$ such that $l^p - l \in K$.

6) a) Let $\alpha \in \bar{\mathbb{Q}}$ be a root of $p(t) := t^4 + 30t^2 + 45$. Show that the extension $L := \mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and find $Gal(L/\mathbb{Q})$,
b) Let $\alpha \in \bar{\mathbb{Q}}$ be a root of $p(t) := t^4+4t^2+2$. Show that the extension $L := \mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and find $Gal(L/\mathbb{Q})$.

7) Let $L \supset K$ be an extension of a prime degree $p, \alpha \in L-K, P(t) := Irr(\alpha, t, K)$. Suppose that there exists $\beta \in L, \beta \neq \alpha$ such that $p(\beta) = 0$. Show that L is a Galois extension of K and find $Gal(L/K)$.